Introduction

1.1 SCALING AND DIMENSIONAL ANALYSIS

The phenomena with which we shall be concerned all exhibit **scaling**. In its simplest form, this just means that two measurable quantities depend upon each other in a **power-law** fashion. A familiar example is Kepler's law, relating the radius R of a planet's circular orbit to the period T of the orbit:

$$T \propto R^{3/2}. \tag{1.1}$$

Another example, possibly not so familiar, is the formula for the phase speed c of waves on shallow water of depth h, neglecting surface tension and viscous effects:

$$c^2 = gh, \tag{1.2}$$

where g is the acceleration due to gravity. The **scaling** law in this example is $c \propto \sqrt{h}$. Formula (1.2) is only valid when the depth is small compared with the wavelength λ, and the more general relation is[1]

$$c^2 = \frac{g\lambda}{2\pi} \tanh\left(\frac{2\pi h}{\lambda}\right), \tag{1.3}$$

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[1] See virtually any text-book on fluid dynamics. A clear presentation is given by D.J. Acheson, *Elementary Fluid Dynamics* (Clarendon, Oxford, 1990), Chapter 3.

which indeed reduces to eqn. (1.2) when $h \ll \lambda$. In this case, then, scaling occurs only approximately, but becomes more and more accurate in the limit $h/\lambda \to 0$.

Often, scaling laws can easily be deduced from **dimensional analysis**. For example, in the water wave case, the only variables that c may depend upon in principle are g, h, λ and the fluid density ρ. Using the notation [ ] to denote the dimensions of a given quantity, and the system of units mass-length-time MLT, we have: $[g] = LT^{-2}$, $[h] = [\lambda] = L$ and $[\rho] = ML^{-3}$ in three spatial dimensions. Thus

$$c = (gh)^{1/2} f \left(\frac{h}{\lambda} \right), \qquad (1.4)$$

where f is a function that cannot be determined by dimensional analysis. In the limit $h/\lambda \to 0$,

$$c \sim (gh)^{1/2} f(0) \propto \sqrt{h}, \qquad (1.5)$$

recovering the scaling law, apparently without requiring any detailed knowledge about fluid dynamics!

Actually, this happy state of affairs is an illusion – we made a very strong assumption in going from eqn. (1.4) to eqn. (1.5), namely *that the limit process was regular*. Usually, this can only be justified properly by considerations other than dimensional analysis. This point is by no means obscure mathematical pedantry: in fact, *the cases where the regularity assumption breaks down constitute the central topic of this book.*

The derivation presented above of the shallow water wave speed is somewhat deceptive for another, more mundane and less far-reaching reason: in writing down eqn. (1.4), we were presented with a choice of which length to use in the prefactor of the function f. We could equally well have used λ instead of h, leading to

$$c = (g\lambda)^{1/2} \tilde{f} \left(\frac{h}{\lambda} \right), \qquad (1.6)$$

where $\tilde{f}$ is another function to be determined. Now what happens in the limit $h/\lambda \to 0$? It looks as if something has gone wrong! To proceed, recall that our purpose in taking the limit is to remove the dependence of c on λ: our common sense intuition tells us that when λ is "sufficiently large" it should not affect the result. The only way that λ can cancel out of the formula (1.6) is if the function $\tilde{f}$ has a square root behaviour for small values of its argument:

$$\tilde{f}(x) \sim x^{1/2} f(x) \qquad \text{as } x \to 0, \qquad (1.7)$$

where the function f is analytic as $x \to 0$ and tends towards some well-defined limit $f(0)$. In fact, f here is the function of eqn. (1.4), as we see by using the approximation (1.7) in eqn. (1.6); we do indeed obtain the correct result (1.5) for the wave speed.

1.2   POWER LAWS IN STATISTICAL PHYSICS

In the above examples, and in many other scaling laws, the power law, or the **exponent** is a rational fraction, often deduced from simple dimensional considerations. This partly accounts for the fact that the phenomena described are so well understood, and are taught in elementary physics courses. However, there is a broader class of phenomena where power-law behaviour occurs, but the exponent is not a simple fraction (as far as is known). This class of phenomena includes, but is by no means restricted to, phase transitions where there is a **critical point**. We shall shortly discuss precisely what we mean by this; but first, let us consider some examples.

1.2.1   *Liquid Gas Critical Point*

In figure (1.1) is sketched a portion of the phase diagram for a fluid. The axes are the temperature T and the density of the fluid ρ, and the curve is shown in the fixed pressure, P, plane. Below the **critical** or **transition temperature**, T_c, is the coexistence curve. This has the following interpretation. Below T_c, as density is increased at fixed temperature, it is not possible to pass from a gaseous phase to a liquid phase without passing through a regime where the container of the fluid contains a mixture of both gas and liquid. The two-phase region has a manifestation in the thermodynamic properties of the fluid, which we will discuss later. Above the critical point, it is possible to pass continuously from a gas to a liquid as the density is increased at constant temperature. In this case, there is no density at which there is a coexisting mixture of liquid and gas in the container. Note that even starting below T_c it is always possible to pass from a liquid to a gas without passing through any two-phase region: one simply raises the temperature above T_c, reduces the density, and then lowers the temperature below T_c. This suggests that there is no real way to distinguish between a liquid and a gas. In fact, the question of how one identifies different phases of matter is one with which we shall be concerned in later chapters.

Returning to figure (1.1), the interesting question to ask for the purposes of the present discussion is: what is the shape of the coexistence
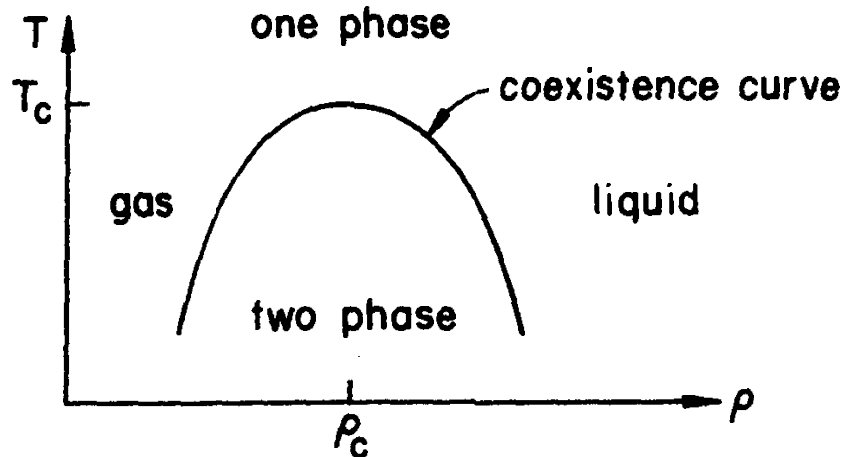
Figure 1.1 Phase diagram of a fluid at fixed pressure.

curve near the critical point? Experimentally, for sulphurhexafluoride, it is found that[2]

$$|\rho_+ - \rho_-| \propto |T - T_c|^{0.327 \pm 0.006} \tag{1.8}$$

is the shape of the curve near the critical point, where $\rho_\pm(T)$ are the values of the density at coexistence on the two branches of the coexistence curve below T_c, as shown in figure (1.1). The number 0.327 ± 0.006 is an example of a **critical exponent**, and does not depend upon the particular fluid system studied. Although you might reasonably have expected that this exponent would be different for the coexistence curve of a different substance, this is not in fact the case! For example, the same measurement on ^{3}He yields a value for the critical exponent[3] of 0.321 ± 0.006. In both of the results quoted, the error bars correspond to two standard deviations[4]. The critical exponent is not obviously a simple rational fraction, and is clearly different from the value $1/2$, which, as we will see later, might have been expected from dimensional analysis. In fact, it was the overwhelming experimental evidence that this exponent was different from $1/2$ that forced some physicists in the 1930's to realise that there was a deep problem lurking in seemingly unimportant exponents.

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[2] The data for the liquid gas critical point of sulphurhexafluoride are taken from M. Ley-Koo and M.S. Green, *Phys. Rev. A* **16**, 2483 (1977).

[3] The ^{3}He data are from C. Pittman, T. Doiron and H. Meyer, *Phys. Rev. B* **20**, 3678 (1979).

[4] A useful summary of the experimental situation is given by J.V. Sengers in *Phase Transitions*, Proceedings of the Cargèse Summer School 1980 (Plenum, New York, 1982), p. 95.

1.2.2 Magnetic Critical Point

A second example is the critical point of a ferromagnet. A magnet may be regarded as consisting of a set of magnetic dipoles residing on the vertices of a crystal lattice. We will often refer to the magnetic dipoles as spins. The spins are able to exchange energy through interactions between themselves, as well as between themselves and other degrees of freedom of the crystal lattice (e.g. via spin-orbit coupling). For systems in equilibrium, one can define a temperature T. If one waits a sufficiently long time, equilibrium is established between the lattice and the spins, and both sets of degrees of freedom are described by a single temperature T. On the other hand, the spins can come into equilibrium between themselves well before they come into equilibrium with the lattice: in this case, the spin degrees of freedom and the lattice degrees of freedom may have different temperatures. Here, we ignore such dynamical questions, and assume that we are dealing with a system described by a single temperature T.

At high temperatures and zero external field, the system is in the **paramagnetic** phase: following the time evolution of any spin would reveal that it points in all directions with equal frequency. Thus, no direction is singled out at any given time when considering all of the spins in the system and the net magnetic moment is zero.

Below a **critical temperature**, T_c, however, the spins tend to align along a particular direction in space, even in the absence of an external field. In this case, there is a net **magnetisation**, $M(T)$, and the system is in the **ferromagnetic** phase. The onset of this behaviour is a **continuous phase transition**: the magnetisation rises continuously from zero as the temperature is reduced below T_c, as sketched in figure (1.2). The magnetisation is zero above the transition and is non-zero below the transition temperature. A quantity which varies in this way is referred to as an **order parameter**.

The question naturally arises as to why the system should order along any particular direction: what is special about the direction? This question is far from being naïve, and we shall discuss it in detail later.

In certain systems, the actual dipole interactions between the atoms on the lattice restrict the spins to point parallel or anti-parallel to one particular direction, which we shall take to be the z-axis. In these systems, known as **Ising ferromagnets** , each spin cannot rotate through all possible orientations, but instead can only point along the $+z$ or $-z$ directions. The Ising ferromagnet is therefore relatively simple to study, and we will devote considerable attention to it in these notes. The simplicity is deceptive, however. What may be simple to state may not be simple to solve!
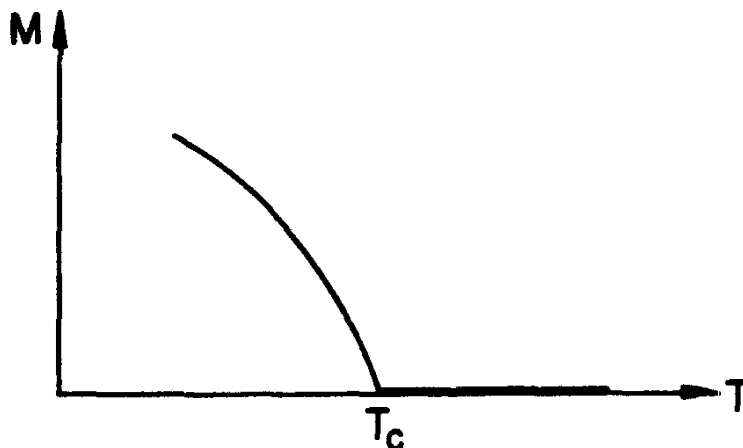
Figure 1.2   Onset of magnetisation in an Ising ferromagnet.

The interaction energy between neighbouring spins in an Ising ferromagnet is lowest when neighbouring spins point in the same direction. However, there is another class of systems, known as Ising **antiferromagnets**, in which the sign of the interaction energy between neighbouring spins is such that the energy is lowered when neighbouring spins point in *opposite* directions. We will later see that the thermodynamics of antiferromagnets with certain crystal lattices in zero applied magnetic field is identical to the thermodynamics of ferromagnets.

The onset of magnetisation in the three dimensional Ising antiferromagnet $DyAlO_3$, in the limit of zero applied magnetic field, exhibits the following behavior experimentally:[5]

$$M \propto (T_c - T)^{0.311 \pm 0.005} \tag{1.9}$$

This result is valid in the limiting case as $T \to T_c$ from below, and is expected to apply to Ising ferromagnets too. As the temperature is reduced below the critical temperature, significant deviations from this result develop. The critical exponent is again not obviously a rational fraction, and furthermore seems to be the same as that for the liquid-gas system, within the experimental precision.

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[5] The experimental results for the critical point of an antiferromagnet were taken from L.M. Holmes, L.G. Van Uitert and G.W. Hull, *Sol. State Commun.* **9**, 1373 (1971). For an exhaustive summary comparing experimental results with predictions based on idealised models of magnetic systems, see L.J. de Jongh and A.R. Miedema, *Adv. Phys.* **23**, 1 (1974).
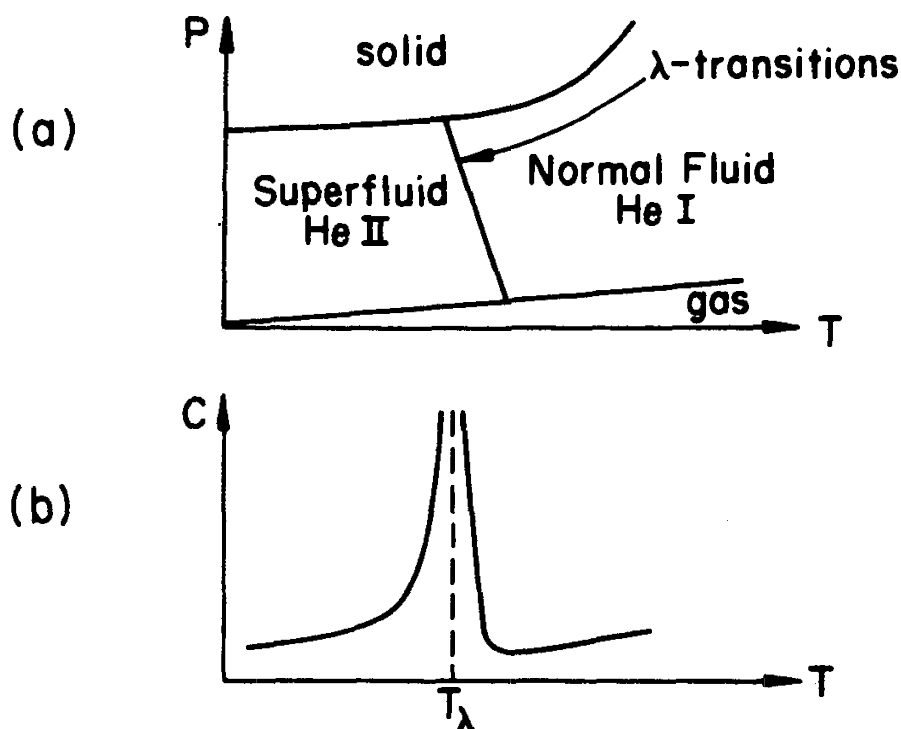
Figure 1.3 (a) Phase diagram of ^{4}He; (b) Heat capacity as a function of temperature at the λ-transition, for fixed pressure.

1.2.3 Superfluid (λ) Transition in ^{4}He

Part of the phase diagram for ^{4}He is sketched in figure (1.3a). For a range of pressures from near zero to about 25 atmospheres, liquid helium undergoes a continuous transition to a superfluid at a temperature of about 2 K. In the superfluid state, ^{4}He exhibits a number of unusual properties, including dissipationless flow through fine capillary tubes. The transition to the superfluid state is sometimes known as the λ-transition, due to the shape of the heat capacity curve, C, as a function of temperature, shown in figure (1.3b). The transition temperature is usually known as T_λ, and its precise value depends upon the pressure.

The best fit to the heat capacity data near the transition[6] is found to be

$$C \propto |T_\lambda - T|^{0.013 \pm 0.003}. \tag{1.10}$$

This is an experiment where great precision is possible for a variety of technical reasons, and there is little doubt that the critical exponent has the *sign* given. This means that the heat capacity curve is actually a cusp, although for many years it was thought that the heat capacity actually exhibited a **divergence at the** λ-transition. Indeed, to a good

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[6] High resolution experiments on the λ-transition are described in J.A. Lipa and T.C.P. Chiu *Phys. Rev. Lett.* **51**, 2291 (1983).

approximation, the behaviour of the heat capacity is logarithmic, and in many books and articles the λ-transition is often described by the formulae

$$C \approx \begin{cases} A\log(T - T_\lambda) + B, & T > T_\lambda \\ A'\log(T_\lambda - T) + B', & T < T_\lambda \end{cases} \tag{1.11}$$

with A, A', B and B' being temperature independent constants. Expressions (1.10) and (1.11) resemble one another when plotted on graph paper over a limited temperature range, as can be seen using the identity

$$x^n = \exp(n\log x) \approx 1 + n\log x \tag{1.12}$$

where the approximation is valid if x is not too small and $n \ll 1$. In the high accuracy experiments leading to the result quoted in equation (1.10), $|T - T_\lambda|/T_\lambda$ ranges from 10^{-3} to 10^{-8}, and deviations from the logarithmic approximation are discernible.

1.2.4   Self-Avoiding Random Walk

Consider the root mean square distance, R, travelled by a random walker after N steps. By root mean square, we imply that an average has been taken over the probability distribution of the walks. Suppose that we now require that the probability distribution does not permit the walk to intersect itself, but otherwise the walks are random. This is sometimes taken to be a minimal model of a polymer chain in solution, because two molecules making up the polymer cannot occupy the same point in space. Such a walk is called a **self-avoiding walk**. In this case, it is found that in three dimensions the simple scaling law for a random walker is changed from $R \propto \sqrt{N}$, and becomes

$$R \propto N^{0.586 \pm 0.004} \tag{1.13}$$

as $N \to \infty$. The claim is that this formula applies to both a real isolated polymer in solution and a mathematical self-avoiding walk. If this is true, then the molecular structure of the polymer and the various energies of interaction between monomers (repeat units) of the polymer do not seem to influence the scaling behaviour. The exponent quoted in equation (1.13) was obtained from experiment on a dilute polymer solution.[7]

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[7] The experimental determination of the scaling of R for polymers is reported in J.P. Cotton, *J. Physique Lett. (Paris)* **41**, L231 (1980). The RG calculations for the same quantity were performed by J.C. Le Guillou and J. Zinn-Justin, *J. Physique Lett. (Paris)* **50**, 1365 (1989).
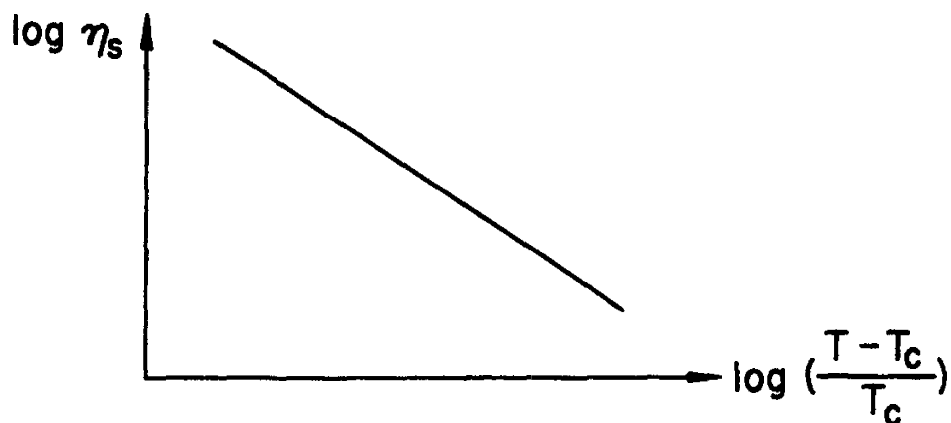
Figure 1.4 Viscosity of a binary fluid near the critical point

Renormalisation group (RG) calculations give a value for the exponent of 0.5880 ± 0.0015, where the uncertainties derive from the mathematical technique used to resum the asymptotic series given by the RG. Indeed, it does seem that the formula is completely independent of the chemistry, and only depends upon the 'spaghetti' nature of the polymer.

1.2.5 Dynamic Critical Phenomena

The examples given above have all exhibited non-trivial power laws in quantities that are unrelated to the time evolution of the physical system in question. For example, magnetisation is a thermodynamic quantity, computed and (in principle) measured in equilibrium. However, non-trivial power laws may also be exhibited by **transport coefficients** in a system near a critical point. A transport coefficient is a phenomenological parameter relating a current to a driving force. For example, Ohm's Law,

$$\mathbf{j} = \sigma \mathbf{E}, \tag{1.14}$$

relates the electric current density $\mathbf{j}$ to the electric field $\mathbf{E}$ which drives the current, through the conductivity tensor σ. Other examples of transport coefficients include diffusion coefficients and viscosity.

Very close to the critical point of a binary fluid mixture, the **shear viscosity**, η_s, is found to diverge weakly with temperature, as sketched in figure (1.4). For example, the shear viscosity of nitroethane-3-methylpentane

is found to behave above T_c as[8]

$$\eta_s \propto \left(\frac{T - T_c}{T_c}\right)^{-0.03 \pm 0.01} . \tag{1.15}$$

This behaviour occurs over a range of temperatures $10^{-5} < (T-T_c)/T_c < 10^{-2}$ where equilibrium thermodynamic quantities also exhibit scaling. Although viscosity is usually thought of as being a property of a flowing fluid, *i.e.* one not in equilibrium, it is generally believed that close to equilibrium, transport properties can be related to *purely equilibrium quantities*. This topic is generally known as **linear response theory**.

1.3   SOME IMPORTANT QUESTIONS

These notes are primarily concerned with phase transitions where there is no generation of latent heat. In other words, I shall, for the most part omit discussion of what are called **first order phase transitions** in the **Ehrenfest classification**. Ehrenfest proposed that phase transitions could be classified as 'n^{th} order' if any n^{th} derivative of the free energy with respect to any of its arguments yields a discontinuity at the phase transition. The phenomena which we will describe are often called **critical phenomena** and occur at **second order phase transitions**, although this name is inappropriate, because the Ehrenfest classification is not correct. I prefer the use of the term **continuous phase transition**. Ehrenfest's classification fails, because at the time that it was formulated, it was not known that thermodynamic quantities such as the specific heat actually *diverge* at continuous transitions, rather than exhibiting a simple **discontinuity**, as the Ehrenfest classification implies. We will see in future chapters that this failure is related to the failure of the applicability of **mean field theory**, as exemplified by the **Weiss theory of ferromagnetism** and the **Van der Waals equation** for fluids.

Power law behavior at a critical point, as in examples (1) and (2) of the previous section, is not just restricted to the quantities $|\rho_+ - \rho_-|$ and M. In fact, we shall see that many different observable quantities exhibit scaling behaviour and have associated **critical exponents**. These quantities can be placed in two categories. The first set of quantities are thermodynamic

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[8] The divergence of the shear viscosity near the critical point was measured by B.C. Tsai and D. McIntyre, *J. Chem. Phys.* **60**, 937 (1974). Further discussion of this topic may be found in the article by J.V. Sengers mentioned in the footnote on page 4. A review on the topic of dynamic critical phenomena is given by P.C. Hohenberg and B.I. Halperin, *Rev. Mod. Phys.* **49**, 435 (1977).

variables, such as the specific heat. The second set characterise spatial ordering in a system and address the question: how does, for example, the *local* magnetic moment vary from point to point in a ferromagnet? The examples which we have mentioned above prompt us to consider many questions, amongst which the most interesting are perhaps:

Question 0: *Why do phase transitions occur at all?*

In statistical mechanics, thermodynamics arises from the free energy, F, and its derivatives, given by Gibbs' formula

$$e^{-F/k_B T} = \text{Tr } e^{-H/k_B T} \tag{1.16}$$

where k_B is Boltzmann's constant and T is the temperature. H is the Hamiltonian, and Tr denotes a sum over all degrees of freedom mentioned in H. Since the Hamiltonian will usually be a non-singular function of the degrees of freedom, the right hand side of equation (1.16) — the **partition function** — is nothing more than a sum of terms, each of which is the exponential of an analytic function of the parameters in the Hamiltonian. How, then, can such a sum give rise to *non-analytic* behaviour, of the sort described in the previous section? Is it even clear *a priori* that a single partition function can describe multiple phases?

Question 1: *How can we calculate the phase diagram of a system as we change the external parameters?*

These external parameters, as well as the temperature, enter the free energy through the Hamiltonian, *via* equation (1.16). Anticipating the answer to the last question posed in the previous paragraph, it is sometimes forgotten that apparently different states of matter, such as liquid and solid, are actually described by the *same* Hamiltonian. Simply changing T in equation (1.16) can cause a change of state.

Question 2: *How do we compute the exponents which are observed at a continuous transition?*

Even when we have understood, in principle, how it is that non-analytic or critical behaviour can arise, the challenge remains to account for the precise values of the critical exponents. Mean field theories, built from the foundations laid by Van der Waals and Weiss, always lead to exponents given by rational fractions. However, the observed values do not seem to agree with these predictions.

The discrepancy between the mean field theory prediction for the shape of the liquid - gas coexistence curve, eqn. (1.8), which gives an

exponent of 1/2, and the observed value of about 0.325 is far from insignificant. Usually in physics, we are satisfied with a qualitative understanding of a given phenomenon together with a reasonable estimate of the quantitative consequences, always with the assurance that a more refined calculation would improve the quantitative predictions. Why then is the difference between 0.5 and 0.35 of such apparent significance? The point is that until about twenty-five years ago or so, *it was not possible, even in principle, to account for this discrepancy.* The numerical discrepancy of 30% in a critical exponent is but the tip of a well-hidden iceberg. Classical physics makes an assumption so subtle that it was not even recognised explicitly for many years. It is no exaggeration to say that in solving this problem, a new way of looking at physics emerged, which has infused condensed matter physics and high energy physics. In recognition of this, K.G. Wilson, the principle architect of the renormalisation group approach, was awarded the 1982 Nobel Prize in physics.

We shall shortly see that the critical exponents are often independent of the specific system under consideration. For example, the critical exponent β for the liquid-gas critical point

$$|\rho_+ - \rho_-| \propto |T - T_c|^{\beta} \tag{1.17}$$

and the exponent β for the ferromagnetic critical point

$$M \propto (T_c - T)^{\beta} \tag{1.18}$$

are the same within the accuracy of the experiments! The fact that two apparently different physical systems might share precisely the same sets of critical exponents is known as **universality**. Thus, we ask

Question 3: *Why does universality occur, and what are the factors that determine which set of phenomena have the same critical exponents?*

Phenomena with the same set of critical exponents are said to form a **universality class**. The usefulness of the concept of universality class lies in the fact that, in general, members of a universality class have only three things in common: the symmetry group of the Hamiltonian (*not* the lattice, if one is present), the dimensionality, and whether or not the forces are short-ranged.

1.4   HISTORICAL DEVELOPMENT

Complete answers to most of the questions above were not known 20 years ago. In these lecture notes, we shall follow approximately the historical development of the subject, although we will start with a precise

statistical mechanical definition of phase transition, emphasising the important notion of the **thermodynamic limit**, in which the system size is taken to be infinitely large, and indicating how, strictly speaking, phase transitions arise only in this limit.

Of central importance to the development of the subject were studies of two physical systems, the liquid-gas system and the ferromagnet, at the *classical* level of description. The term 'classical' is not used to indicate an alternative to quantum theory; instead, it signifies that the theory in question ignores thermal fluctuations, an approximation used in all but a few exactly solvable cases, until the advent of the renormalisation group. In fact, we shall see that these classical theories, due to Van der Waals and Weiss respectively, are actually rather good in many ways, and already exhibit some (but not all) qualitatively correct features of phase transitions. In our presentation, we will also expose some of the similarities between these two apparently different phenomena, showing how the mathematical descriptions of these phenomena become identical near a critical point.

This observation forms the basis of **Landau's theory of phase transitions**, which is the most succinct encapsulation of the classical approach. However, we will present Landau theory in a way which anticipates the developments that follow. In fact, we shall see that classical theories, and thus Landau's theory, are **mean field theories**; a physical variable such as the magnetisation is replaced by its average value, and fluctuations about that value are ignored. It is possible to use Landau theory to estimate the importance of fluctuations, and thus to check on the self-consistency of the theory. However, it is found that near a critical point, fluctuations are not negligible, and thus the theory is not self-consistent. Landau theory contains within it the seeds of its own destruction!

The next significant step was the development of the notion of **scaling laws**, first arrived at on a phenomenological basis by B. Widom. We will see that the equation of state of a physical system near to a critical point obeys what appears to be an analogue of the **law of corresponding states**, encountered in the equation of state (such as that due to Van der Waals) for a fluid. In the latter case, however, the law of corresponding states is always valid, not just near to a critical point. In the context of magnetic systems, the equation of state relates the magnetisation, M, the temperature T, and the **external magnetic field**, H:

$$H = f(M, t), \qquad t = \left| \frac{T - T_c}{T_c} \right| \tag{1.19}$$

Ostensibly, f is a function of two variables. Nevertheless, Widom discovered that near a critical point, the equation of state may be written as

$$H = M^\delta \Phi \left(\frac{t}{M^{1/\beta}} \right) \qquad (1.20)$$

with Φ being a function of just *one* variable. The exponents δ and β appear in equation (1.20) in accord with convention, and Widom's striking discovery was that δ, β and T_c can be chosen so that the experimental data from *different* materials (Fe, Ni, ...) all satisfy equation (1.20) *with the same function* Φ! Furthermore, it was noticed that equation (1.20) implies a relationship between the critical exponents for different thermodynamic quantities, such as the specific heat, the susceptibility and the magnetisation.

Where do scaling laws come from? L.P. Kadanoff proposed a simple, intuitive explanation, namely that a relation of the form of (1.20) follows if one assumes that near a critical point, the system 'looks the same on all length scales'. We shall formulate this notion a bit more precisely later and see why it is, in fact, not really quite correct. Kadanoff's argument is important, because it provides the basic physical insight on which the technique of the **renormalisation group (RG)** is built; Kadanoff almost certainly was aware of the flaws in his argument, but nevertheless had the intellectual courage to propose it anyway.

The modern era began with a series of seminal papers by K.G. Wilson in 1971, in which the renormalisation group was developed and explained in the contexts of both condensed matter physics and high energy physics. These and subsequent papers by Wilson initiated an explosion of activity which continued unabated for a decade. Many of the applications of the renormalisation group utilised perturbation expansion techniques, with such small parameters as the variables $\epsilon \equiv 4 - d$, where d is the dimensionality of space and $1/n$, where n is the number of components of the order parameter (*i.e.*, the magnetisation, which is a vector, has 3 components). Today, some of this body of work has become part of the mainstream of physics.

At the time of writing, it is probably fair to say that the frontiers of renormalisation group physics have shifted away from phase transitions and field theory, towards **non-equilibrium phenomena.** One active avenue of research is the study of dynamical phenomena where there is a fluctuating noise source present. An example is the growth of an interface by the random deposition of atoms. Such an interface is rough, and will exhibit height fluctuations about its average position. The RG and other

methods, originally developed to treat critical phenomena, have been used to study how these fluctuations scale in both space and time.

Another line of inquiry is the dynamics of systems approaching an equilibrium state, but still not close enough to equilibrium that linear response theory is valid. In these systems, spatial correlations sometimes exhibit scaling behaviour in time. One well-known example is that of the **phase separation** of a binary alloy below the critical point, where the time dependent X-ray scattering intensity at wavenumber k, $S(k,t)$, is found to obey a relation of the form

$$S(k,t) = t^{d\phi}F(kt^{\phi}) \tag{1.21}$$

with ϕ close, if not exactly equal, to $1/3$, and d being the spatial dimensionality. It is not yet well-understood how to apply the RG to pattern formation problems of this type.

In another set of problems described by partial differential equations without noise present, a system may be well-described by a **similarity solution** of the form

$$u(x,t) \sim t^{\alpha}f(xt^{\beta}), \qquad \text{as } t \to \infty, \tag{1.22}$$

where u is some observable and x and t are space and time respectively. In some cases, the exponents α and β characterising the similarity solution may not be simply obtained by dimensional analysis; nevertheless, the RG can be successfully used to solve these problems too. Examples of such problems arise in many areas of fluid mechanics, for example, and this new topic will be introduced in chapter 10.

The condensed matter physics literature contains two versions of the RG: the Gell-Mann–Low RG and the Wilson RG. In critical phenomena, the use of the former is based upon perturbation theory, whilst the latter has a direct *geometrical* interpretation and is in principle, non-perturbative. Indeed, it was the introduction of Wilson's method in 1971 which began the modern era of the RG. The connection between the two versions is by no means obvious. We shall present the Gell-Mann–Low RG when we explain renormalisation in chapter 10, in the context of phenomena far from equilibrium; there we will also mention the connection with Wilson's formulation of RG. We will see that renormalisation has a direct physical interpretation, and may be easily understood without the technical complications of quantum field theory.

EXERCISES

Exercise 1-1

Dimensional analysis (DA) is often a powerful tool in physics. This question requires you to use dimensional arguments to solve a couple of interesting problems. DA is usually used in two ways: (1) The fundamental theorem of DA asserts[9] that in any physical problem involving a number of dimensionful quantities, the relationship between them can be expressed by forming all possible independent dimensionless quantities, denoted by Π, Π_1, Π_2, ..., Π_n. Then the solution to the physical problem is of the form $\Pi = f(\Pi_1 \ldots \Pi_n)$, where f is a function of n variables. (2) Sometimes there is only one dimensionless combination of variables relevant to a given problem. Then (1) implies $\Pi =$ constant.

(a) By noting that the area of a right-angled triangle can be expressed in terms of the hypotenuse and (*e.g.*) the smaller of the acute angles, prove Pythagoras' theorem using dimensional analysis. You will find it helpful to construct a well-chosen line in the right-angled triangle. *Note: the whole point of dimensional analysis is that you do NOT need to solve for the functional form of the solution to a given problem. Thus, in this question, you must pretend that you do not know trigonometry.*

(b) Now consider the case of Riemannian or Lobachevskian geometry (*i.e.* the triangle is drawn on a curved surface such as a riding saddle or a football). What happens in this case?

Exercise 1-2

In 1947, a sequence of photographs of the first atomic bomb explosion in New Mexico in 1945 were published in *Life* magazine. The photographs show the expansion of the shock wave caused by the blast at successive times in ms. From the photographs, one can read off the radius of the shock wave as a function of time: the result is shown in the accompanying table. Assuming that the motion of the shock is unaffected by the presence of the ground, and that the motion is determined only by the energy released in the blast E and the density of the undisturbed air into which the shock is propagating, ρ, derive a scaling law for the radius of the fireball as a function of time. Use the data from the photographs to test your scaling law and hence deduce the yield of the blast. *You must test*

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[9] First enunciated apparently by E. Buckingham in a delightful paper (see especially the concluding paragraph), *Phys. Rev.* **4**, 345 (1914). Fourier is usually attributed with the principle that every term in a physical equation must have the same dimensions.

Table 1.1 RADIUS R OF BLAST WAVE AFTER TIME T

T/msec	R/m
0.10	11.1
0.24	19.9
0.38	25.4
0.52	28.8
0.66	31.9
0.80	34.2
0.94	36.3
1.08	38.9
1.22	41.0
1.36	42.8
1.50	44.4
1.65	46.0
1.79	46.9
1.93	48.7
3.26	59.0
3.53	61.1
3.80	62.9
4.07	64.3
4.34	65.6
4.61	67.3
15.0	106.5
25.0	130.0
34.0	145.0
53.0	175.0
62.0	185.0

your scaling law by plotting a graph. You should consider carefully and then explain what is the most useful graph to plot. You should assume that all numerical factors are of order unity![10] Although the photographs were declassified in 1947, the yield of the explosion was to remain classified until several years later.

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[10] For a detailed analysis and discussion of the data, see G.I. Taylor, *Proc. Roy. Soc. A* **201**, 175 (1950).