that is, the parallel straight lines $y = 2tx + C$, which become continually steeper as time goes on. From the equations of the paths, we see that the fluid is moving like a rigid body, keeping its orientation, and its points describing congruent parabolas.

Exercises.

1. Study the motions

a) $x = \frac{x_0 + y_0}{2} e^t + \frac{x_0 - y_0}{2} e^{-t}, \quad y = \frac{x_0 + y_0}{2} e^t - \frac{x_0 - y_0}{2} e^{-t}, \quad z = z_0,$

b) $x = x_0 + \sin t, \quad y = y_0 + (1 - \cos t), \quad z = z_0,$

c) $\frac{dx}{dt} = x, \quad \frac{dy}{dt} = y, \quad \frac{dz}{dt} = z,$

determining the nature of the paths, the velocity fields, and the lines of flow.

2. Show by a simple example that, in general, the path of a particle, moving under a stationary field of force, will not be a line of force.

4. Expansion, or Divergence of a Field.

An important concept in connection with a fluid in motion is its rate of expansion or contraction. A portion of the fluid occupying a region T_0 at time t_0, will, at a later time t, occupy a new region T. For instance, in the steady flow of the last section, a cylinder bounded at $t = 0$ by the planes $z_0 = 0$, $z_0 = 1$, and by the surface $x_0^2 + y_0^2 = a^2$, becomes at the time t the cylinder bounded by the same planes and the surface

$$\frac{x^2}{(a e^t)^2} + \frac{y^2}{(a e^{-t})^2} = 1,$$

as we see by eliminating x_0, y_0, z_0 between the equations of the initial boundary and the equations of the paths (fig. 6). Here the volume of the region has not changed, for the area of the elliptical base of the cylinder is πa^2, and so, independent of the time.

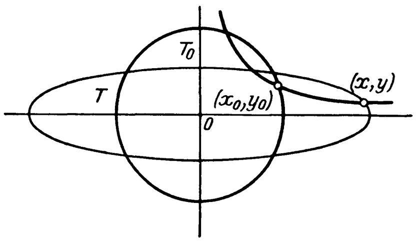

Fig. 6.

On the other hand, in the flow

$$x = x_0 + t, \quad y = y_0 e^t,$$

the same cylinder at time $t = 0$, has at the time t the elliptical boundary

$$\frac{(x-t)^2}{a^2} + \frac{y^2}{(a e^t)^2} = 1,$$

so that the volume has increased to $\pi a^2 e^t$. The time rate of expansion of this volume is the derivative of this value, also $\pi a^2 e^t$. If we divide the rate of expansion of the volume by the volume, and find such a quotient for a succession of smaller and smaller volumes containing a given point, the limit gives us the time rate of expansion per unit of volume at that point. In the present instance, the quotient is 1, und by decreasing a, we may make the original volume as small as we please. Hence the time rate of expansion per unit of volume at the point originally at the origin is always 1. It is not hard to see that this characterizes the rate of expansion of the fluid at all points, for the chords of any portion of the fluid parallel to the x- and z-axes are constant, while those parallel to the y-axis are increasing at the relative rate 1. Thus every cubic centimetre of the fluid is expanding at the rate of a cubic centimetre per second.

Let us now consider the rate of expansion in a general flow. The volume at time t is

$$V(t) = \iiint_T dx\,dy\,dz\,.$$

We must relate this expression to the volume at t_0. By the equations (2), every point (x, y, z) of T corresponds to a point (x_0, y_0, z_0) of T_0. We may therefore, by means of this transformation, in which t is regarded as constant, change the variables of integration to x_0, y_0, z_0. According to the rules of the Integral Calculus[1], this gives

$$V(t) = \iiint_T dx\,dy\,dz = \iiint_{T_0} J(x_0, y_0, z_0, t)\,dx_0\,dy_0\,dz_0\,,$$

where J denotes the Jacobian, or functional determinant

$$J(x_0, y_0, z_0, t) = \begin{vmatrix} \frac{\partial x}{\partial x_0}, & \frac{\partial y}{\partial x_0}, & \frac{\partial z}{\partial x_0} \\ \frac{\partial x}{\partial y_0}, & \frac{\partial y}{\partial y_0}, & \frac{\partial z}{\partial y_0} \\ \frac{\partial x}{\partial z_0}, & \frac{\partial y}{\partial z_0}, & \frac{\partial z}{\partial z_0} \end{vmatrix}$$

of the transformation.

We are interested in the time rate of expansion of the volume. This is given, if the Jacobian has a continuous derivative with respect to the time, by

$$\frac{dV}{dt} = \iiint_{T_0} \frac{dJ}{dt}\,dx_0\,dy_0\,dz_0\,.$$

We can compute the derivative of the Jacobian for $t = t_0$ without difficulty, and as t_0 can be taken as any instant, the results will be general. First,

$$\frac{dJ}{dt} = S \begin{vmatrix} \frac{\partial^2 x}{\partial t\,\partial x_0}, & \frac{\partial^2 y}{\partial t\,\partial x_0}, & \frac{\partial^2 z}{\partial t\,\partial x_0} \\ \frac{\partial x}{\partial y_0}, & \frac{\partial y}{\partial y_0}, & \frac{\partial z}{\partial y_0} \\ \frac{\partial x}{\partial z_0}, & \frac{\partial y}{\partial z_0}, & \frac{\partial z}{\partial z_0} \end{vmatrix}$$

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[1] See Osgood: *Advanced Calculus*, New York, 1925, Chap. XII, §§ 4—8, or Courant: *Differential- und Integralrechnung*, Berlin, 1927—29, Vol. II, pp. 261, 264.

where the symbol S means that we are to add two more determinants in which the second and third rows of J, instead of the first, have been differentiated with respect to t. Let us assume that all derivatives appearing are continuous. Then, since x, y, z reduce to x_0, y_0, z_0, for $t = t_0$, at this instant

$$\frac{\partial x}{\partial x_0} = \frac{\partial y}{\partial y_0} = \frac{\partial z}{\partial z_0} = 1, \quad \frac{\partial x}{\partial y_0} = \frac{\partial x}{\partial z_0} = \frac{\partial y}{\partial x_0} = \frac{\partial y}{\partial z_0} = \frac{\partial z}{\partial x_0} = \frac{\partial z}{\partial y_0} = 0,$$

$$\frac{\partial^2 x}{\partial t\,\partial x_0} = \frac{\partial}{\partial x_0}\left(\frac{\partial x}{\partial t}\right) = \frac{\partial X}{\partial x_0}, \quad \frac{\partial^2 y}{\partial t\,\partial y_0} = \frac{\partial Y}{\partial y_0}, \quad \frac{\partial^2 z}{\partial t\,\partial z_0} = \frac{\partial Z}{\partial z_0}.$$

Accordingly

$$\left.\frac{dJ}{dt}\right]_{t=t_0} = \left.\frac{\partial X}{\partial x_0} + \frac{\partial Y}{\partial y_0} + \frac{\partial Z}{\partial z_0}\right]_{t=t_0}.$$

We may now drop the subscripts, since x, y, z coincide with x_0, y_0, z_0, at $t = t_0$, and t_0 may be any time. We then have, for the time rate of expansion of the fluid occupying a region T at time t,

$$\text{(6)} \qquad \frac{dV}{dt} = \iiint_T \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) dV.$$

From this equation we may derive the relative rate of expansion, or the rate of expansion per unit of volume at a point. We remove the integrand from under the sign of integration, by the law of the mean, and divide by the volume:

$$\frac{\frac{dV}{dt}}{V} = \overline{\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}}.$$

If, now, the region T is made to shrink down on the point $P(x, y, z)$, the limit of the above expression gives us the *relative time rate of expansion of the fluid at P*:

$$\text{(7)} \qquad \operatorname{div} \boldsymbol{V} = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z},$$

or the *divergence* of the vector field $\boldsymbol{V}(X, Y, Z)$, as it is called. The expression (6) is called the *total divergence* of the field for the region T.

We see at once that if the rate of change of volume (6) is everywhere 0, the divergence (7) is everywhere 0, and conversely. Thus a fluid whose divergence vanishes everywhere is *incompressible*[1].

We are now in a position to see how the field lines can give us a picture of the intensity of the field. Consider all the field lines passing through a small closed curve. They generate a tubular surface called a *field tube*, or, in a field of force, a *tube of force*. If the flow is stationary,

[1] See, however, § 9 (p. 45).

the fluid flows in this tube, never crossing its walls. If, in addition, the fluid is incompressible, it must speed up wherever the tube is pinched down, and slow down when the tube broadens out. Interpreting the field as a field of force, we see that *in a stationary field of force whose divergence vanishes everywhere, the force at the points of a line of force is greater or less according as the neighboring lines of force approach or recede from it.* This qualitative interpretation of the spacing of the lines of force will be made more exact in § 6.

Exercises.

1. Verify that the field of Exercise 1, page 31, has a divergence which vanishes everywhere. Draw the lines of force $3x^2y - y^3 = C$ for $C = -2, -1, 0, 1, 2$, and verify the relationship between intensity and spacing of the field lines.

2. Verify the fact that the total divergence vanishes for the field of force due to a single particle, for regions not containing the particle, bounded by conical surfaces with the particle as vertex, and by concentric spheres. Show that for spheres with the particle at the centers, the total divergence is $-4\pi m$, where m is the mass of the particle.

3. A *central field of force* is one in which the direction of the force is always through a fixed point, and in which the magnitude and sense of the force depends only on the distance from the point. The fixed point is called the *center* of the field. Show that the only field of force with Q as center, continuous except at Q, whose divergence vanishes everywhere except at Q, is the Newtonian field of a particle at Q. Thus Newton's law acquires a certain geometrical significance.

4. An *axial field of force* is one in which the direction of the force is always through a fixed line, and in which the magnitude and sense of the force depends only on the distance from this line. The line is called the *axis* of the field. If such a field is continuous, and has a vanishing divergence everywhere except on the axis, find the law of force. Find also the law of force in a field with vanishing divergence in which the force is always perpendicular to a fixed plane and has a magnitude and sense depending only on the distance from this plane.

5. Show that the divergence of the sum of two fields (the field obtained by vector addition of the vectors of the two fields) is the sum of the divergences of the two fields. Generalize to any finite sums, and to certain limits of sums, including integrals. Thus show that the divergence of Newtonian fields due to the usual distributions vanishes at all points of free space.

6. The definition of the divergence as

$$\lim_{V \to 0} \frac{\frac{dV}{dt}}{V}$$

involves no coördinate system. Accordingly, the expression (7) should be independent of the position of the coördinate axes. Verify that it is invariant under a rigid motion of the axes.

5. The Divergence Theorem.

The rate of expansion of a fluid can be computed in a second way, and the identity obtained by equating the new and old expressions will be of great usefulness. Let us think of the fluid occupying the region