

$\sigma_B$  are the mean values of  $\sigma$  on the two sublattices, one can use (24.70) to calculate them self-consistently. For example, the mean value of any variable  $\sigma_{\vec{R}_A}$  is

$$\sigma_A = \langle \sigma_{\vec{R}_A} \rangle = \frac{e^{\{\beta\mu_B H + \beta J z \sigma_B\}} - e^{\{-\beta\mu_B H - \beta J z \sigma_B\}}}{e^{\{\beta\mu_B H + \beta J z \sigma_B\}} + e^{\{-\beta\mu_B H - \beta J z \sigma_B\}}}. \quad \text{The probability that } \sigma_{\vec{R}_A} = 1 \text{ times 1 plus the probability it equals } -1 \text{ times } -1. \quad (24.71)$$

Thus

$$\sigma_A = \tanh[\beta\mu_B H + \beta z \sigma_B J] \quad (24.72a)$$

$$\sigma_B = \tanh[\beta\mu_B H + \beta z \sigma_A J]. \quad (24.72b)$$

The chemical potential  $\mu$  is determined by the requirement that atoms  $A$  and  $B$  be present in equal proportion, which implies that

$$\sigma_A + \sigma_B = 0. \quad \begin{array}{l} \text{If } N_A \text{ and } N_B \text{ are the numbers of } A \text{ and } B \\ \text{sublattice sites, then } \sum_{\vec{R}} \sigma_{\vec{R}} = N_B - N_A = \\ \sum_{\vec{R}_A} \sigma_{\vec{R}} + \sum_{\vec{R}_B} \sigma_{\vec{R}} = N_A \sigma_A + N_B \sigma_B. \end{array} \quad (24.73)$$

One can only achieve Eq. (24.73) if  $\mu_B H = \mu - z(\epsilon_{BB} + \epsilon_{AB})/2 = 0$ , so finally one has

$$\sigma_A = -\tanh(\beta J z \sigma_A) = \tanh(\beta |J| z \sigma_A). \quad \text{Remember that superlattice ordering only happens when } J = \epsilon_{AB} < 0. \quad (24.74)$$

Because Eq. (24.74) is identical to Eq. (24.56), it can be solved by the method indicated in Figure 24.7.

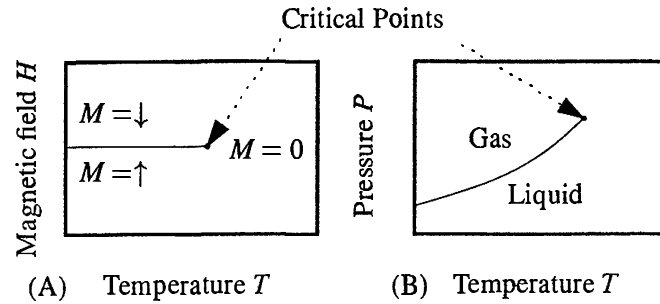
### 24.5.2 Spin Glasses

Spin glasses were discovered by Jacobs and Schmitt (1959). They occur when the interactions between spins have random sign or magnitude. There are typically huge numbers of nearly degenerate states separated by energy barriers of many different sizes. Edwards and Anderson (1975) wrote down the analog of the Ising model for these systems, and it was solved by Parisi (1987).

## 24.6 Critical Phenomena

Mean field theory is a crude approximation, but it describes the physics of phase transitions rather well, except near one special location on the phase diagram, the *critical point*, sketched in Figure 24.10. What makes this point so critical? As particularly emphasized by Fisher (1974) a comparison of phenomena applicable to magnets and to fluids provides some explanation.

- The critical point is a unique location in the phase diagram. In a magnetic system, the critical point occurs for zero external magnetic field and at the critical temperature  $T_c$ , which is the lowest temperature at which the spontaneous magnetization vanishes. In a fluid, for any given pressure there is a temperature below which fluid and gas can coexist, and above which fluid and gas merge into a single phase. The critical point occurs at the pressure for which this critical temperature is as high as possible. In  $\text{CO}_2$ , for example, the critical point is at 72 atm and a temperature of  $31.04^\circ\text{C}$ .



**Figure 24.10.** (A) Schematic phase diagram for a ferromagnet. The critical point lies at the highest temperature where there is spontaneous magnetization in zero field. (B) Schematic phase diagram of liquid–gas system. Beyond the critical point, fluid and gas phases merge and cannot be distinguished as pressure varies. The two phases of the magnet are related by symmetry, but this is not true in the liquid–gas system.

- The specific heat both of magnets and fluids diverges approaching the critical point. The divergence takes the form of a power law.
- The magnetic susceptibility diverges in magnetic systems, and the compressibility diverges in fluids. These divergences also take the forms of power laws.
- The divergences result from large fluctuations: large correlated domains of spins flipping back and forth in magnets, and large regions altering between one density and another in fluids. Fluids that normally are transparent become milky, displaying *critical opalescence*.

Investigating these phenomena led to two important ideas:

**Universality.** Divergences near the critical point are identical in a variety of apparently different physical systems and also in a collection of simple models. Systems group into a small number of *universality classes*. For example, ferromagnetic salts, carbon dioxide, and the Ising model all behave identically near the critical point, and belong to the same class. However, two-dimensional magnetic films are essentially different from three-dimensional magnetic systems, and they belong to a different class.

**Scaling.** The key to understanding the critical point lies in understanding the relationship between systems of different sizes. Scaling functions, such as those used to describe localization in Section 18.5.2, are the key to encoding the universal features of the critical point. Formal development of this idea led to the *renormalization group* of Wilson (1975).

### 24.6.1 Landau Free Energy

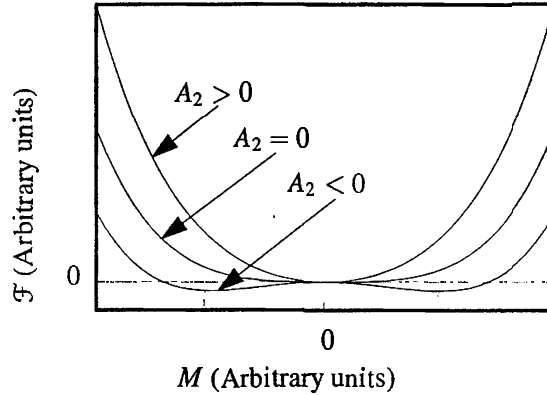
In order to see that mean field theory fails near the critical point, it is necessary to analyze its predictions and compare them with experiment. This task could be carried out by starting with Eq. (24.56). However, it is preferable to adopt a

framework developed by Landau (1937) that exhibits clearly the minimal set of assumptions needed to obtain the results of mean field theory.

Suppose that the free energy of a magnetic system is *analytic* in the vicinity of the critical point, and that all physical quantities can be expressed as a Taylor series in the magnetization and temperature. This assumption seems reasonable, because the magnetization vanishes right at the critical point, and one only needs to know how the free energy depends upon it when it is small. So

$$\mathcal{F}(M, T) = A_0(T) + A_2(T)M^2 + A_4(T)M^4 + HM. \quad \text{The coupling between the small magnetization } M \text{ and the external field } H \text{ is given by Eq. (24.26).} \quad (24.75)$$

The coefficients  $A_0(T) \dots A_4(T)$  are undetermined functions of temperature. The reason that only even terms in  $M$  appear is that the system must be invariant under  $M \rightarrow -M, H \rightarrow -H$ ; terms odd in  $M$  alone must vanish.



**Figure 24.11.** Form of Landau's free energy, Eq. (24.75), for  $A_2 > 0, A_2 = 0,$  and  $A_2 < 0.$  Energies are measured relative to  $A_0.$

Let  $T_c$  be the critical temperature, and define the *reduced temperature*

$$t \equiv \frac{T - T_c}{T_c}, \quad (24.76)$$

which is a dimensionless variable designed to vanish right at the critical temperature. Phase separation begins when  $\mathcal{F}$  develops a concave structure.  $A_4$  must always be positive or else  $\mathcal{F}$  is minimized by sending  $M$  to infinity.  $A_0$  sets the zero of energy and has little significance. Therefore the onset of nonzero magnetization is governed by  $A_2(T)$ . As shown in Figure 24.11, the shape of  $\mathcal{F}$  changes when  $A_2$  passes through zero, so the critical point must be the place where  $A_2$  vanishes. Measure energies relative to  $A_0(T_c)$ , and let  $a_2$  and  $a_4$  be constants. The form of the free energy in the vicinity of the critical point must be

$$\mathcal{F} = a_2 t M^2 + a_4 M^4 + HM. \quad (24.77)$$

**Spontaneous Magnetization.** The equilibrium magnetization is determined by minimizing the free energy. Therefore, it must satisfy

$$H + 2ta_2M + 4a_4M^3 = 0. \quad (24.78)$$

When  $H = 0$ ,  $M = 0$  always satisfies Eq. (24.78). However, for  $t < 0$  there are two more solutions:

$$M = \begin{cases} \pm \sqrt{\frac{2|t|a_2}{4a_4}} & \text{for } t < 0 \\ 0 & \text{for } t > 0. \end{cases} \quad (24.79)$$

Figure 24.11 shows that for  $t < 0$  these values of  $M$  give a lower value of the free energy than  $M = 0$ , and therefore correspond to equilibrium.

**Specific Heat.** With the behavior of  $M$  in hand it is possible to evaluate a collection of other quantities. The specific heat is determined by the identity

$$C_V = \frac{\partial \mathcal{E}}{\partial T} = \frac{\partial}{\partial T} \frac{\partial \beta \mathcal{F}}{\partial \beta} \quad (24.80)$$

For a derivation from thermodynamics, see for example, Landau and Lifshitz (1980) pp. 47–48. The relation for  $\mathcal{E}$  also is easily derived by writing out the expression for  $\mathcal{E}$  in the canonical ensemble.

$$= -\frac{1}{T_c} \frac{\partial}{\partial t} (1+t)^2 \frac{\partial}{\partial t} \left( \frac{\mathcal{F}}{1+t} \right) \quad \text{Use Eq. (24.76).} \quad (24.81)$$

$$\approx -\frac{1}{T_c} \frac{\partial^2 \mathcal{F}}{\partial t^2} \quad \text{Focusing on small } t. \quad (24.82)$$

$$= \begin{cases} \frac{1}{T_c} \frac{a_2^2}{2a_4} & \text{for } t < 0 \\ 0 & \text{for } t > 0. \end{cases} \quad \text{Insert Eq. (24.79) into Eq. (24.77).} \quad (24.83)$$

**Magnetic Susceptibility.** Turning on an external field  $H$  changes the equilibrium magnetization. For  $t < 0$ , take

$$M = \sqrt{\frac{2|t|a_2}{4a_4}} + qH, \quad (24.84)$$

where  $q$  is to be determined. Placing Eq. (24.84) into Eq. (24.78) and expanding to first order in  $H$ , one finds that

$$q = -\frac{1}{4a_2|t|}. \quad (24.85)$$

Expanding similarly about  $M = 0$  for  $t > 0$  gives

$$\frac{\partial M}{\partial H} \approx \begin{cases} -\frac{1}{4|t|a_2} & \text{for } t < 0 \\ -\frac{1}{2ta_2} & \text{for } t > 0. \end{cases} \quad (24.86)$$

**Critical Isotherm.** Finally, right on the critical isotherm at  $t = 0$ , one has that

$$H + 4a_4M^3 = 0 \Rightarrow M \propto H^{1/3}. \quad (24.87)$$

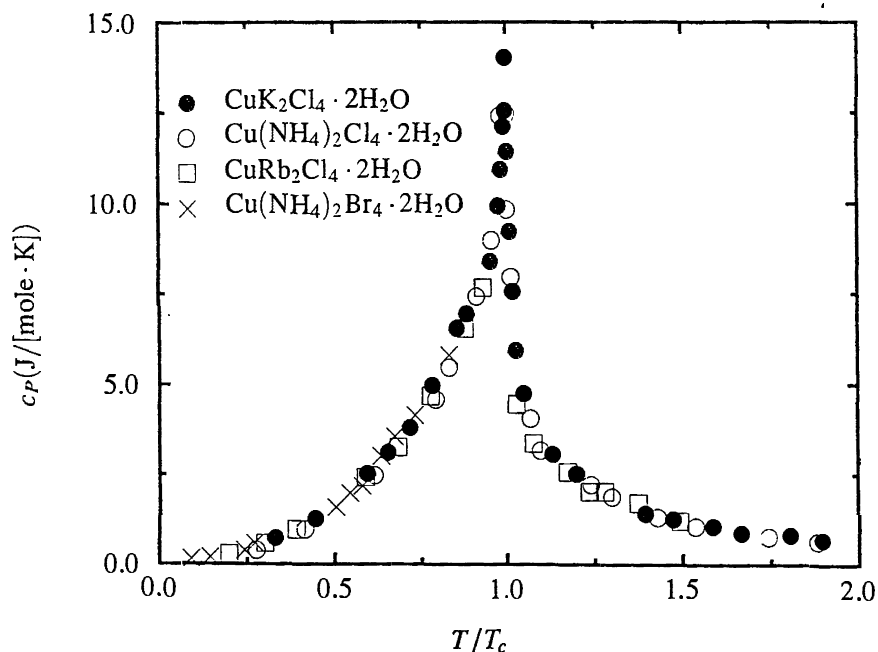
The suppositions of the Landau theory are entirely reasonable, yet experiments prove them wrong. The assumption that fails is the assumption that the free energy can be expanded in a Taylor series about the critical point.

**Correspondence between Liquids and Magnets.** Experiments on critical point phenomena have been conducted both in magnets, and in liquid–gas systems. The two systems act almost identically in the vicinity of the critical point, once corresponding variables have been identified.

A correspondence between magnetic and liquid–gas systems can be constructed in a manner similar to that used for superlattices in Section 5.2.3. Imagine that the material of type *A* in that section is a fluid, while the material of type *B* is gas. Then the equations describing the superlattice can immediately be interpreted as describing liquid–gas phase transitions. At temperatures below the transition, fluid prefers to group with fluid and gas prefers to group with gas, meaning that the energy  $\epsilon_{AB}$  is positive, and the fluid–gas system corresponds to a ferromagnet, with positive  $J$  in Eq. (24.43). Above the transition temperature, fluid and gas mix together, forming a single homogeneous phase, while below the transition, gas and liquid phase separate.

Based on this discussion, one forms the following correspondence: The magnetization  $M$  corresponds to the difference in particle density  $\Delta n$  between liquid and gas. The thermodynamic variable conjugate to  $M$  is the magnetic field  $H$ , so the variable conjugate to  $\Delta n$ , which is the chemical potential  $\mu$ , must correspond to magnetic field. The chemical potential is hard to measure directly. However, according to the Gibbs–Duhem relation

$$dP = sdT + nd\mu, \quad \text{See Landau and Lifshitz (1980), p. 72; } s \text{ is the entropy per volume.} \quad (24.88)$$



**Figure 24.12.** Molar heat capacities of four ferromagnetic copper salts versus scaled temperature  $T/T_c$ . [Source Jongh and Miedema (1974).]

so that if temperature is fixed, pressure is proportional to chemical potential, and pressure can therefore be used instead of chemical potential as the analog of magnetic field.

Six exponents are conventionally defined to characterize various quantities that become singular near the critical point.

**Specific Heat— $\alpha$ .** The specific heat both in fluids and in magnets diverges as

$$C_V(t) \sim |t|^{-\alpha}; \quad (24.89)$$

data for magnets appear in Figure 24.12. Mean field theory predicts a discontinuity in the specific heat, not a divergence.

**Magnetization and Density— $\beta$ .** Figure 24.13(A) displays experimental measurements of temperature versus magnetization near the critical point. According to Eq. (24.79),  $T$  should approach  $T_c$  as  $M^2$ . The data however show that  $T$  approaches  $T_c$  as  $M^3$ . Figure 24.13(B) shows that temperature versus density at constant pressure for a collection of liquid–gas systems is characterized by the same exponent. So

$$M \sim |t|^\beta \quad \text{and} \quad \Delta n \sim |t|^\beta. \quad (24.90)$$

Letting  $n_c$  be the density at the critical point, one can take  $\Delta n$  to be  $n_{\text{liquid}} - n_c$ ,  $n_c - n_{\text{gas}}$ , or  $n_{\text{liquid}} - n_{\text{gas}}$ .

**Compressibility and Susceptibility— $\gamma$ .** The isothermal compressibility of fluids diverges near the critical point:

$$K_T = \frac{1}{n} \frac{\partial n}{\partial P} \sim \frac{1}{n_c} \frac{\partial \Delta n}{\partial P} \sim |t|^{-\gamma}. \quad (24.91)$$

The analogous divergence for a magnet is

$$\frac{\partial M}{\partial H} = \chi \sim |t|^{-\gamma}. \quad (24.92)$$

Mean field theory predicts  $\gamma = 1$ , but the measurements find a slightly larger exponent.

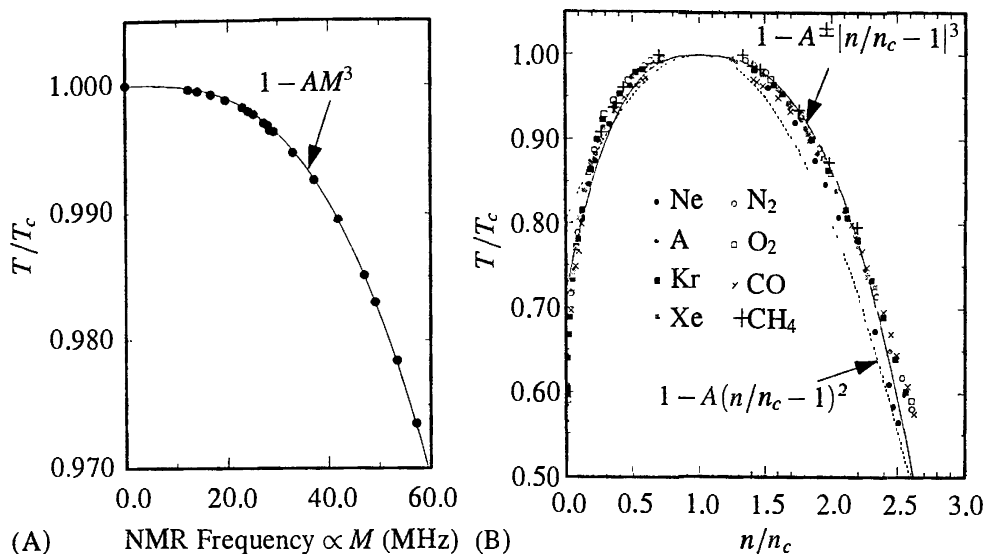
**Critical Isotherm— $\delta$ .** A next critical exponent occurs by making measurements right at  $T_c$  on the critical isotherm

$$P \sim |\Delta n|^\delta, \quad (24.93)$$

and for a magnet

$$|M| \sim |H|^{1/\delta}. \quad (24.94)$$

The mean field prediction is  $\delta = 3$ , but the measurements find  $\delta \approx 5$ .



**Figure 24.13.** (A) Temperature versus magnetization, measured using nuclear magnetic resonance by Heller and Benedek (1962) for the antiferromagnet  $MnF_2$ , near the critical temperature  $T = 67.336$  K. The data are fit well by  $T/T_c = 1 - A|M|^3$ . (B) Coexistence curve for eight fluids, measured by Guggenheim (1945). The data can be fit reasonably well by curves of the form  $T/T_c = 1 - A^\pm |n/n_c - 1|^3$ , where  $A^\pm$  is a constant taking different values depending on whether  $n$  is greater or less than  $n_c$ . The best quadratic fit is also shown, but it suits the data less well.

**Correlation Length —  $\nu$ .** A final pair of exponents relates to light scattering experiments that probe the correlation function of fluids. According to Eq. (5.39) the two-particle correlation function  $g$  is closely related to the scattering function  $S$ . Near the critical point,  $g(r)$  is observed to behave as

$$g(r) - 1 \sim e^{-r/\xi} \quad \text{There are power-law corrections given in Eq. (24.99),} \quad (24.95)$$

so  $\exp[-r/\xi]/r^{1+\eta}$  is more accurate.

where  $\xi(T)$  is called the *correlation length*. The implication for scattering experiments is that

$$S(\vec{q}) - 1 = n \int d\vec{r} e^{i\vec{q}\cdot\vec{r}} [g(r) - 1] \quad (24.96)$$

$$\sim \int d\vec{r} e^{-r/\xi + i\vec{q}\cdot\vec{r}} \sim \frac{1}{1 + \xi^2 q^2}. \quad (24.97)$$

The correlation length  $\xi$  diverges approaching the critical point, with

$$\xi \sim |t|^{-\nu}. \quad (24.98)$$

This divergence can be measured by plotting  $1/S(q)$  versus  $q^2$

**Power-Law Decay at Critical Point— $\eta$ .** Right at the critical point, the correlation function  $g$  decays as a power and not as an exponential. The rate at which it decays is

$$g(r) \sim r^{-1-\eta}, \quad (24.99)$$

**Table 24.3.** Summary of critical exponents, showing correspondence between fluid-gas systems, magnetic systems, and the three-dimensional Ising model

Exponent	Fluid	Magnet	Mean Field Theory	Experiment	3d Ising
$\alpha$	$C_V \sim  t ^{-\alpha}$	$C_V \sim  t ^{-\alpha}$	discontinuity	0.11–0.12	0.110
$\beta$	$\Delta n \sim  t ^\beta$	$M \sim  t ^\beta$	$\frac{1}{2}$	0.35–0.37	0.325
$\gamma$	$K_T \sim  t ^{-\gamma}$	$\chi \sim  t ^{-\gamma}$	1	1.21–1.35	1.241
$\delta$	$P \sim  \Delta n ^\delta$	$ H  \sim  M ^\delta$	3	4.0–4.6	4.82
$\nu$	$\xi \sim  t ^{-\nu}$	$\xi \sim  t ^{-\nu}$		0.61–0.64	0.63
$\eta$	$g(r) \sim r^{-1-\eta}$	$g(r) \sim r^{-1-\eta}$		0.02–0.06	0.032

Source: Vicentini-Missoni (1972) p. 67, Cummins (1971), p. 417, and Goldenfeld (1992) p. 384.

and this equation defines the exponent  $\eta$ .

Table 24.3 presents a summary of the six exponents and gives their values in mean field theory and experiment.

### 24.6.2 Scaling Theory

The Landau theory of phase transitions makes no assumption more severe than that the free energy could be expanded as a Taylor series in the neighborhood of the critical point. Because the Landau theory fails, this assumption must fail, and the question is what should replace it.

A full theory of the critical point is extremely elaborate. The concept of universality makes it possible to construct a theory by focusing upon simple models, such as the Ising model, whose critical behavior is identical to that of the experimental systems. Yet even for these model systems, the analysis is extremely involved. The most accurate determination of critical behavior for the Ising models is given by power-series expansions of the partition function, discussed by Domb (1974).

Rather than developing the theory, the discussion will focus upon developing the language needed to describe experimental observations. The basic observation is that near the critical point, physical quantities behave as power laws, that these power laws are universal, but that the exponents are far from obvious. It has become commonplace to claim that natural phenomena behave as power laws, but the standard set in the field of critical phenomena, where precise power-law scaling is observed over many decades, has rarely been matched.

Widom (1965) showed that many experimental observations could be described by assuming a particular type of scaling relation between thermodynamic variables. The scaling theory can take a number of different forms, one of which is to make an hypothesis about the manner in which variables appear within the free energy.

Consider the power-law divergence of the specific heat, shown in Figure 24.12. Because the specific heat is obtained from the free energy as in Eq. (24.80), the free energy  $\mathcal{G}(T, H)$  must have a singular piece. Separate the singularity out in the



form

$$\frac{\mathcal{G}}{\mathcal{V}k_B T} = |t|^{x_1} G(t, H), \tag{24.100}$$

where the exponent  $x_1$  has been chosen so that when  $H = 0$ ,  $G$  is a smooth function of  $t$ . At  $H = 0$  the specific heat must diverge as  $|t|^{-\alpha}$ , so one has that

$$C_V = \frac{\partial}{\partial T} \frac{\partial \beta \mathcal{G}}{\partial \beta} \sim t^{-\alpha} \tag{24.101}$$

$$\Rightarrow x_1 = 2 - \alpha. \tag{24.102}$$

Notice that differentiating by  $T$  lowers the power of  $t$ , but multiplying by  $T$  does not raise the power of  $t$ ; it multiplies by the constant  $T_c$ .

**Scaling Form for Free Energy.** The main difficulty posed by a form such as Eq. (24.101) lies in the experimental fact that when  $H \neq 0$ , there are no singular quantities as  $t$  passes through zero. The specific heat is only singular right at the critical point. With a nonzero magnetic field, the specific heat displays a peak as a function of temperature, but varies smoothly. Choosing a form for  $G$  in accord with this experimental observation proceeds in two steps.

First, assume that the magnetic field and temperature enter  $G$  only in the combination

$$G(t, H) = G\left(\frac{H}{H_0 |t|^\Delta}\right). \text{ } H_0 \text{ is some constant, and } \Delta \text{ is some exponent.} \tag{24.103}$$

One way to express the idea behind this functional form is to say that the importance of the magnetic field to the free energy can only be judged by comparing it to some reference value, and it is supposed that this reference value scales as a power law with the reduced temperature.

Second, assume that  $G$  itself behaves as a power when its argument becomes extremely large:

$$\lim_{y \rightarrow \infty} G(y) \sim y^{x_2}. \tag{24.104}$$

Making this assumption, one has for small  $|t|$  but nonzero  $H$  that

$$\frac{\mathcal{G}}{\mathcal{V}k_B T} \sim |t|^{2-\alpha} \left(\frac{H}{H_0 |t|^\Delta}\right)^{x_2} \sim |t|^{2-\alpha-\Delta x_2}. \tag{24.105}$$

Thus the free energy can be made nonsingular whenever  $H \neq 0$ , provided that

$$x_2 = \frac{2 - \alpha}{\Delta}. \tag{24.106}$$

The exponent  $\Delta$  can be expressed entirely in terms of exponents that have already been defined by examining the spontaneous magnetization. One has

$$-M = \frac{\partial \mathcal{G}}{\partial H} = |t|^{2-\alpha} \frac{1}{H_0 |t|^\Delta} G' \left(\frac{H}{H_0 |t|^\Delta}\right). \text{ See Eq. (24.31).} \tag{24.107}$$

When  $H \rightarrow 0$ , the magnetization must vanish as  $|t|^\beta$ , so

$$|t|^{2-\alpha-\Delta} \sim |t|^\beta \quad (24.108)$$

$$\Rightarrow \Delta = 2 - \alpha - \beta. \quad (24.109)$$

Notice one peculiarity. Above the critical temperature, the spontaneous magnetization must vanish, yet it is predicted to have the same power law divergence above as below. These two facts are consistent if one observes that the coefficients of the power-law can be different above and below  $T_c$ : the coefficient above  $T_c$  is just zero.

**Relations Among Exponents.** Having determined the singular parts of the free energy in terms of the exponents  $\alpha$  and  $\beta$ , it is possible to continue calculating the various singular quantities that are found experimentally. All the remaining singularities can now be related to the ones that have already been found.

For example, the magnetic susceptibility is

$$\left. \frac{\partial M}{\partial H} \right|_{H=0} = \chi \sim \frac{|t|^{2-\alpha}}{H_0^2 |t|^{2\Delta}} G'' \left( \frac{H}{H_0 |t|^\Delta} \right) \Big|_{H=0} \quad (24.110)$$

$$\Rightarrow |t|^{2-\alpha-2\Delta} \sim |t|^{-\gamma} \quad (24.111)$$

$$\Rightarrow \gamma = \alpha + 2\Delta - 2. \quad (24.112)$$

Combining Eq. (24.112) with Eq. (24.109) gives

$$2 = \alpha + 2\beta + \gamma. \quad (24.113)$$

This relation among exponents, the *Widom relation*, is a consequence of the scaling assumption, and it is obeyed in all known cases.

The exponent  $\delta$  describes the relation of the magnetization to the magnetic field on the critical isotherm. As  $t \rightarrow 0$ ,

$$M \sim \frac{1}{H_0 |t|^\Delta} |t|^{2-\alpha} \left( \frac{H}{H_0 |t|^\Delta} \right)^{x_2-1} \quad (24.114)$$

$$\sim H^{x_2-1} = H^{(2-\alpha-\Delta)/\Delta} \quad (24.115)$$

$$\Rightarrow \frac{1}{\delta} = \frac{2-\alpha-\gamma}{2-\alpha+\gamma} \quad (24.116)$$

$$\Rightarrow \delta = 1 + \frac{\gamma}{\beta}, \quad \text{Use Eq. (24.113).} \quad (24.117)$$

which is the *Rushbrooke relation*.

The final exponents,  $\nu$  and  $\eta$ , relate to properties of the correlation function  $g(r)$ , so the connection between the correlation function and the free energy needs to be determined. The relation is provided by observing that fluctuations in particle number are related to the compressibility  $K_T$  through

$$\langle \Delta N^2 \rangle = -\frac{k_B T N^2}{\mathcal{V}^2} \frac{\partial \mathcal{V}}{\partial P} = k_B T n^2 \mathcal{V} K_T \quad \text{See Landau and Lifshitz (1980), p. 342.} \quad (24.118)$$

$$= \left[ \int d\vec{r} d\vec{r}' \langle n(\vec{r})n(\vec{r}') \rangle \right] - \langle N \rangle^2 \quad (24.119)$$

$$= \mathcal{V}n \left\{ 1 + n \int d\vec{r} (g(r) - 1) \right\}. \quad (24.120)$$

Follows steps similar to those  
around Eq. (5.31), and use  
Eq. (5.36) for  $g$ .

Because near the critical point

$$g(r) \sim \frac{e^{-r/\xi}}{r^{1+\eta}}, \quad (24.121)$$

one has

$$K_T \sim \int d\vec{r} g(r). \quad (24.122)$$

Changing variables to  $\vec{s} = \vec{r}/\xi$  and using Eq. (24.121), one obtains

$$K_T \sim \xi^3 \xi^{-1-\eta} \int d\vec{s} \frac{e^{-s}}{s^{1+\eta}} \quad (24.123)$$

$$\sim \xi^{2-\eta} \sim |t|^{-\nu(2-\eta)}. \quad (24.124)$$

However, the compressibility must diverge as  $|t|^{-\gamma}$ , so

$$(2-\eta)\nu = \gamma, \quad (24.125)$$

the *Fisher-Essam relation*. Finally, there is an apparently improbable argument that because  $G = \mathcal{G}/k_B T \mathcal{V}$  has dimensions of inverse length to the third power, but the only length in the system that should be important near the critical point is the correlation length  $\xi$ , one must have that

$$\frac{\mathcal{G}}{k_B T \mathcal{V}} \sim |t|^{2-\alpha} \sim \xi^{-3} \quad (24.126)$$

$$\Rightarrow 2 - \alpha = 3\nu, \quad (24.127)$$

the *Josephson or hyperscaling relation*. All of these scaling relations are obeyed within a few percent by the experimental values listed in Table 24.3.

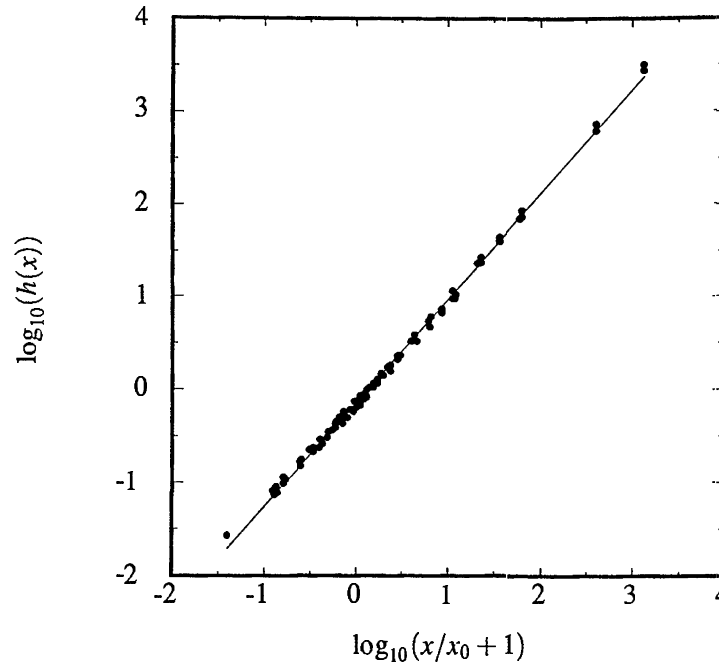
**Scaling Form for Magnetization.** Some of the most detailed experimental results are expressed using an alternate form of the scaling hypothesis, one that relates the magnetic field and magnetization through

$$|H| = |M|^\delta h \left( \frac{t}{|M|^{1/\beta}} \right). \quad (24.128)$$

The function  $h$  can be measured by choosing temperature  $T$  and field  $H$ , measuring  $M$ , and then constructing the ratio  $|H|/|M|^\delta$  and plotting it versus the variable

$$x = \frac{t}{|M|^{1/\beta}}. \quad (24.129)$$

When the external magnetic field  $H$  vanishes,  $h(x)$  must vanish. Because  $|M| \sim t^\beta$  for  $H = 0$ , the conclusion is that in vanishing field  $x = t/|M|^{1/\beta} = -x_0$  is a constant, and  $h(-x_0) = 0$ .



**Figure 24.14.** Log–log plot of scaling function  $h = |H|/|M|^\delta$  versus  $x = t/|M|^{1/\beta}$ , using  $\delta = 4.32$ ,  $\beta = 0.364$ , and  $x_0 = 0.596$  for  $\text{CrBr}_3$ . The exponents and the critical temperature  $T_c = 32.841$  K are determined together as part of the process of trying to ensure that data taken at different temperatures lie on top of a single scaling curve. [Source: Vicentini-Missoni (1972), p. 68.]

If the scaling hypothesis is correct, then measurements of  $|H|/|M|^\delta$  versus  $x + x_0$  should fall on a single line. As shown in Figure 24.14, the data do collapse in this way, and furthermore the function  $h$  takes the form of a power law over a large range of temperature near the critical point.

### Problems

1. **Magnetic dipole moment:** Consider a small loop of wire in the  $x$ – $y$  plane with area  $A$ , and current  $J$  flowing through it. Show that the vector potential  $\vec{A}$  far from the loop is given by

$$\vec{A} = \frac{\vec{m} \times \vec{r}}{r^3}, \quad (24.130)$$

where  $\vec{r}$  is a vector from the middle of the loop of wire to an observation point.

2. **Magnetic dipole energies:** Verify Eqs. (24.36) and (24.37):

- (a) Consider a distribution of current  $\vec{j}(\vec{r})$  that is localized and is in steady state, which means that  $\vec{\nabla} \cdot \vec{j} = 0$ . By considering

$$\int d\vec{r} r_\alpha r_\beta \vec{\nabla} \cdot \vec{j} = 0, \quad (24.131)$$